

Resource Estimation for Matrix Inversion via QSVT with the Clifford+T Gate Set

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Introduction

Solving the linear system of equations $Ax = b$ is one of the problems where quantum computers are expected to have an advantage over classical computers. The time and space complexity of existing classical algorithms, even for matrices with relatively good properties, is at best $O(\text{poly}(N))$ with respect to the matrix size $N = \max(N_{\text{row}}, N_{\text{col}})$ [14, 15]. This limits the size of problems that can be addressed by classical algorithms. Owing to the progress in recent quantum algorithm research, initiated by the pioneering work by Harrow et al. [6], an $O(\text{polylog}(N))$ complexity has been achieved for certain classes of matrices. However, there is still a lack of quantitative evaluation of computational cost based on explicit gate implementations. To bridge this gap, we constructed a quantum circuit for a specific problem using the standard universal gate set, Clifford+T gates, and numerically evaluated its accuracy and computational cost.

Backgrounds

Given a signal $x \in [-1, 1]$, there is a known method called Quantum Signal Processing (QSP) that achieves polynomial transformations of x by alternately applying a signal operator $W(x)$ that encodes the signal and a parametric signal processing operator $S(\phi)$;

$$W(x) := \begin{pmatrix} x & i\sqrt{1-x^2} \\ i\sqrt{1-x^2} & x \end{pmatrix} = e^{i \arccos(x)X}, \quad (1)$$

$$S(\phi) := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} = e^{i\phi Z}. \quad (2)$$

For any polynomials $P(x), Q(x) \in \mathbb{C}[x]$ that satisfy certain conditions (see [5] for detail), there exists a sequence of angles $\Phi = (\phi_0, \phi_1, \dots, \phi_d) \in \mathbb{R}^{d+1}$ such that the following equation holds for $x \in [-1, 1]$ [8, 5]

$$S(\phi_0) \prod_{k=1}^d W(x) S(\phi_k) = \begin{pmatrix} P(x) & iQ(x)\sqrt{1-x^2} \\ iQ^*(x)\sqrt{1-x^2} & P^*(x) \end{pmatrix} \quad (3)$$

QSP can be extended to multi-qubit system by the method called Quantum Singular Value Transformation (QSVT) [5, 9]. With QSVT, when a matrix A is given in the form of a block-encoding, we can obtain a block-encoding of the matrix whose singular values $\{\sigma_i\}$ are transformed by the real part of the polynomial P .

In particular, if an appropriate polynomial P that approximates the function $1/x$ is given, it is possible to compute an approximate inverse of A by

$$P^{(\text{SV})}(A)^\dagger \simeq W_\sigma \begin{pmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_N} \end{pmatrix} V_\sigma^\dagger = A^{-1}. \quad (4)$$

Here, W_σ and V_σ are two unitary matrices whose columns are left and right singular vectors of A . Now we can obtain the solution to the linear system of equations as follows:

$$\begin{pmatrix} P^{(\text{SV})}(A) & \cdot \\ \cdot & \cdot \end{pmatrix}^\dagger \begin{pmatrix} b \\ 0 \end{pmatrix} \simeq \begin{pmatrix} A^{-1}b \\ \cdot \end{pmatrix} = \begin{pmatrix} x \\ \cdot \end{pmatrix}. \quad (5)$$

In the next section, we briefly summarize the methods we adopt for each step of the matrix inversion using QSVT.

Methods

Matrix

In this study, we consider Toeplitz matrices, whose elements along each diagonal are constant. Due to its regular pattern, block-encoding circuit can be efficiently constructed using rotation gates and adders [13]. Specifically, let N_{data} be the number of non-zero elements in the matrix, then the block-encoding of a Toeplitz matrix can be realized using N_{data} rotation gates and $4 \log_2 N$ Toffoli gates. If N_{data} is $O(\log(N))$, the total number of gates, including other Clifford gates, is $O(\text{polylog}(N))$.



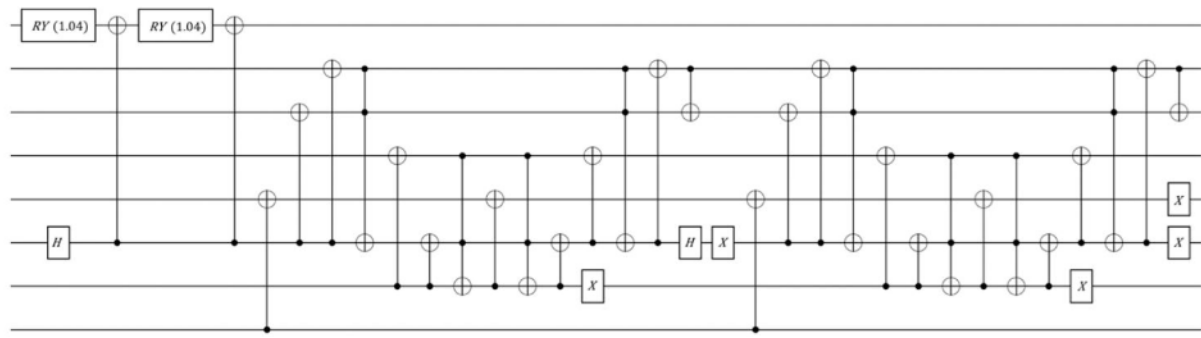


Figure 1: A quantum circuit for block-encoding of Toeplitz matrix ($N = 4$).

●●● In this study, we consider a simple matrix

$$A = \begin{pmatrix} 2 & 1 & & \\ & 2 & \ddots & \\ & & \ddots & 1 \\ & & & 2 \end{pmatrix}, \quad (6)$$

and explicitly implement the corresponding block-encoding circuit using the method in [13]. As an example, the block-encoding circuit for ($N = 4$) is shown in Fig. 1. Here, we use the method by Möttönen et al. [10] for data loading, and adopt two ripple carry adders [3] to compile the additions appearing in eq. (55) of [13].

We note that the condition number of this specific matrix asymptotically approaches 3 as the matrix size grows.

Polynomial approximation and angle finding

Several methods have been proposed to obtain a polynomial that approximates the function $1/x$ over a closed interval $D_{\kappa\text{QSVT}} = [1/\kappa\text{QSVT}, 1]$, given the degree d of the polynomial. We utilize the explicit form of the optimal polynomial P and the complementary polynomial Q , proposed by [2]. Since the condition number of the matrix can be considered almost constant as mentioned in Section 3.1, we fix $\kappa\text{QSVT} = 4$ for the following experiments. We then adopt the carving algorithm [5] to compute the corresponding QSP angle sequence Φ .

Decomposition of non-Clifford gates

To explicitly evaluate the computational cost, we decompose the non-Clifford gates included in the constructed QSVT circuit into the Clifford+T gate set. In Table 1, we summarize the non-Clifford gates appearing in our implementation. The rotation gates are approximately decomposed into a sequence of H, S and T gates using the Ross-Selinger algorithm [11]. The multi-controlled gates are decomposed using doubly-controlled $\pm iX$ gates, following [12]. Toffoli gates in ripple carry adders [3] used for block-encoding are decomposed into Clifford+T gates via the method in [1].

By adding ancillary qubits, further reductions in T-count can be achieved. For example, the T-count of multi-controlled gates and ripple carry adders can approximately be halved by using «temporary logical-AND» structure (relative phase Toffoli gate) [4]. For data loading, there is a known trade-off between the number of ancillas and T-count [7]. While this paper does not delve into the specifics, these are important considerations for practical implementation.

Results

We numerically evaluated the implemented quantum circuit in terms of accuracy and computational cost.

Non-Clifford gates	# of appearance	T -count after decomp.	# of ancilla
R_Z for QSP	d	$\sim 3 \log_2 \frac{1}{\epsilon_{\text{gate}}}$	0
R_Y for data loading	$N_{\text{data}} d$	$\sim 3 \log_2 \frac{1}{\epsilon_{\text{gate}}}$	0
Multi-controlled gate	d	$8(\log_2 N_{\text{data}} + 1)$	$\log_2 N_{\text{data}}$
Adder in block-encoding	$2d$	$14 \log_2 N$	0

Table 1: Non-Clifford Gates.

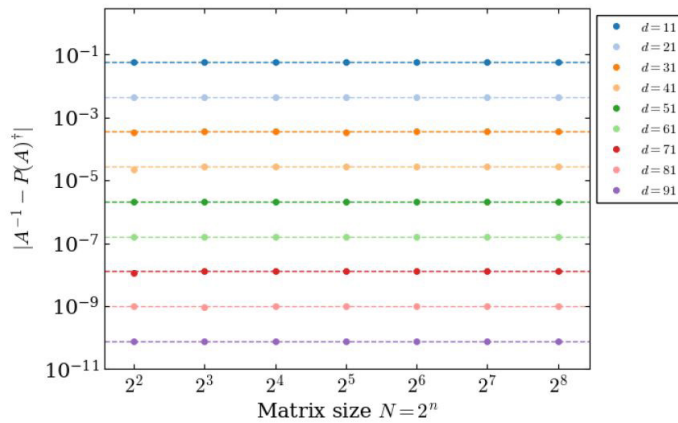


Figure 2: Dependence of the error on the size of the matrix and the degree of the polynomial.

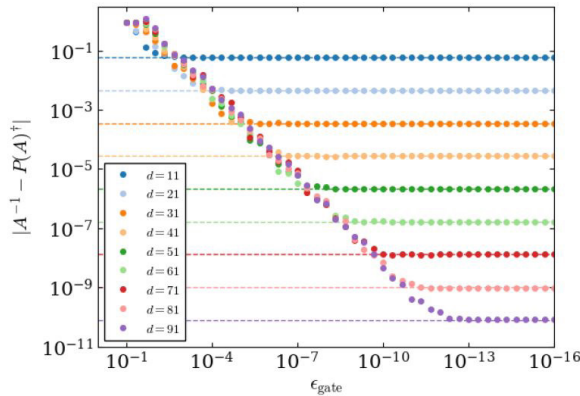


Figure 3: Dependence of the error on the degree of the polynomial and the precision for decomposition of rotation gates.

Accuracy

First, we investigated the dependence of the output accuracy (evaluated by the operator norm $\|A^{-1} - P^{(SV)}(A)^{\dagger}\|$) on the degree d of the approximate polynomial and the matrix size $N = 2n$. As shown in Fig. 2, we confirmed that the output accuracy improves exponentially with respect to d , and is independent of N .

Next, we fixed the matrix size at $N = 8$ and decomposed the non-Clifford gates. Fig. 3 shows the dependency of the output accuracy on the decomposition error ϵ_{gate} of the rotation gates for different d . In the left region of the figure, where ϵ_{gate} is large, the decomposition error of the rotation gates is dominant. On the other hand, in the right region where ϵ_{gate} is sufficiently small, the error due to polynomial approximation (indicated by the dotted

lines in the figure) becomes dominant, and the output accuracy of the quantum circuit saturates.

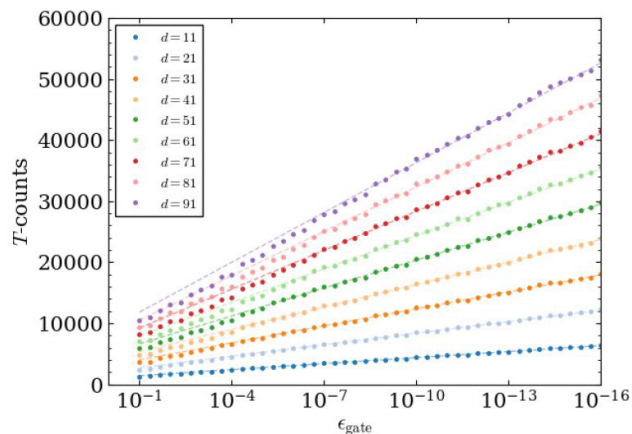


Figure 4: Dependence of the T-count on the degree of the polynomial and the precision for decomposition of rotation gates.

The error due to the decomposition of the rotation gates is considered to be proportional to the product of ϵ_{gate} and the number of rotation gates $(N_{data}+1)d$. Therefore, when each rotation gate is decomposed with an accuracy of ϵ_{gate}' and a final output accuracy of ϵ is required, the degree d of the approximate polynomial is determined by the minimum singular value of the scaled matrix A and some proportional constant. This, in turn, allows us to determine the computational cost discussed in the next section.

Computational cost

We evaluated the computational cost in regards of the number of qubits and the T-count. It was found that the number of logical qubits n_{qubits} is explicitly given as

$$n_{qubits} = 2 \log_2 N + \log_2 N_{data} + 6, \quad (7)$$

which does not depend on the decomposition error gate of the rotation gates or the degree d of the approximation polynomial. Next, we examined the behavior of the T-count arises from the decomposition of non-Clifford gates. As can be seen from Fig. 4, it is confirmed that the T-count with respect to d and ϵ_{gate} matches the expectations outlined in Table 1 (indicated by the dotted lines in the figure).



Summary

We explicitly constructed a quantum circuit to compute the inverse of Toeplitz matrices via QSVT with the Clifford+T gate set, and evaluated its accuracy and computational cost. As a result, we confirmed:

- (i) The output accuracy of the inverse matrix improves exponentially with respect to the degree d of the approximate polynomial.
- (ii) The output accuracy does not depend on the matrix size N , since the condition number of the matrix is bounded by a constant in our specific choice of matrix.
- (iii) The T-count depends on d and the decomposition error of the rotation gates ϵ_{gate} as shown in Table 1, and it depends logarithmically on N through the adder in the block-encoding circuit.

In order to construct an end-to-end quantum linear system solver, the preparation of the input quantum states and the readout of the output states must also be implemented.

It is known that both, if implemented naively, may require exponential cost, which would undermine quantum advantage. Addressing this issue remains a challenge for future work. ■

Abstract

Solving linear systems of equations is a fundamental problem that serves as a crucial foundation in numerous fields. Initiated by the pioneering Harrow-Hassidim-Lloyd algorithm, the asymptotic performance of quantum linear system solvers has steadily improved overtime. However, there remains a lack of quantitative evaluation of quantum resources based on explicit gate implementations. In this study, we particularly focus on the matrix inversion method via quantum singular value transformation and explicitly construct a quantum circuit to invert specific matrices using the standard universal gate set, Clifford+T gates. We examine the detailed methods for block-encoding, QSP angle finding, and gate decompositions, and numerically evaluate the overall resources required for the quantum circuit. ■

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